A Proof of the Fundamental Theorem of Calculus

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Despite being such an important theorem for the theory of integration, the fundamental theorem of calculus (FTC) is fairly straightforward to prove. In this note, we prove a theorem that is usually considered as one half of the full FTC and is the main result used for the evaluation of integrals.

We use the Darboux integral, which is equivalent to the usual Riemann integral. The main tool of the proof is the mean value theorem.

Theorem. *Let a and b be real numbers with $a < b$, and let f and F be real-valued functions on $[a, b]$. Suppose that f is bounded and integrable and that F is continuous on $[a, b]$ and differentiable on (a, b). If $F'(x) = f(x)$ for all $x \in (a, b)$, then*

$$\int_a^b f(x)\, dx = F(b) - F(a). \tag{1}$$

Proof. Let $P = \{x_0, \ldots, x_n\}$ be a partition of $[a, b]$, where n is a positive integer and

$$a = x_0 < x_1 < \cdots < x_n = b. \tag{2}$$

For each $k \in \{1, \ldots, n\}$, since F is continuous on $[x_{k-1}, x_k]$ and differentiable on (x_{k-1}, x_k), by the mean value theorem there exists a $c_k \in (x_{k-1}, x_k) \subseteq (a, b)$ such that

$$F(x_k) - F(x_{k-1}) = F'(c_k)(x_k - x_{k-1}). \tag{3}$$

We have that $F'(c_k) = f(c_k)$ since $c_k \in (a, b)$, and so

$$F(x_k) - F(x_{k-1}) = f(c_k)(x_k - x_{k-1}). \tag{4}$$

Next, define

$$M_k = \sup\{f(x) : x \in [x_{k-1}, x_k]\} \quad \text{and} \quad m_k = \inf\{f(x) : x \in [x_{k-1}, x_k]\}. \tag{5}$$

Then,

$$M_k(x_k - x_{k-1}) \geq f(c_k)(x_k - x_{k-1}) = F(x_k) - F(x_{k-1}) \tag{6}$$

and

$$m_k(x_k - x_{k-1}) \leq f(c_k)(x_k - x_{k-1}) = F(x_k) - F(x_{k-1}). \tag{7}$$

Hence,

$$U(f,P) = \sum_{k=1}^{n} M_k(x_k - x_{k-1}) \geq \sum_{k=1}^{n} F(x_k) - F(x_{k-1}) \tag{8}$$

and

$$L(f,P) = \sum_{k=1}^{n} m_k(x_k - x_{k-1}) \leq \sum_{k=1}^{n} F(x_k) - F(x_{k-1}). \tag{9}$$

We have the telescoping sum[1]

$$\sum_{k=1}^{n} F(x_k) - F(x_{k-1}) = F(x_n) - F(x_0) = F(b) - F(a), \tag{10}$$

and thus

$$L(f,P) \leq F(b) - F(a) \leq U(f,P). \tag{11}$$

Since P was arbitrary,

$$\underline{\int_a^b} f(x)\, dx = \sup_P L(f,P) \leq F(b) - F(a) \leq \inf_P U(f,P) = \overline{\int_a^b} f(x)\, dx. \tag{12}$$

By assumption, f is integrable, and so

$$\underline{\int_a^b} f(x)\, dx = \overline{\int_a^b} f(x)\, dx = \int_a^b f(x)\, dx \tag{13}$$

and therefore

$$\int_a^b f(x)\, dx = F(b) - F(a). \tag{14}$$

□

[1]It is interesting to note that the telescoping of a sum is a discrete version of the FTC, where we replace integrals with sums and derivatives with differences. This “discrete” FTC then shows up in the proof of the “continuous” FTC.