# Redundancy in the Axioms for a Topology

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The definition of a topology on a set is usually given as follows.

**Definition.** Let $X$ be a set. A topology $\mathcal{T}$ on $X$ is a collection of subsets of $X$ such that

1. $\varnothing, X \in \mathcal{T}$,
2. $\mathcal{T}$ is closed under arbitrary unions, and
3. $\mathcal{T}$ is closed under finite intersections.

However, it turns out that the first axiom of a topology is redundant: if we assume that $\mathcal{T}$ is closed under arbitrary unions and finite intersections, we can derive that $\varnothing$ and $X$ are both in $\mathcal{T}$.

To show this, we first need to recall the definition[1] of the union or intersection of a collection of subsets of a given set.

**Definition.** Let $X$ be a set and $\mathcal{B}$ be a collection of subsets of $X$. The union of the sets in $\mathcal{B}$ (relative to $X$) is defined as

$$\bigcup \mathcal{B} = \{x \in X : x \in B \text{ for some } B \text{ in } \mathcal{B}\} \tag{1}$$

and the intersection of the sets in $\mathcal{B}$ (relative to $X$) is defined as

$$\bigcap \mathcal{B} = \{x \in X : x \in B \text{ for all } B \text{ in } \mathcal{B}\}. \tag{2}$$

With these definitions, we can compute the union and intersection of an empty collection of subsets.

**Proposition.** [2] *Let $X$ be a set. Then, relative to $X$,*

$$\bigcup \varnothing = \varnothing \tag{3}$$

*and*

$$\bigcap \varnothing = X. \tag{4}$$

[1] Some authors define arbitrary unions and intersections differently so as to explicitly require the collection of subsets to be nonempty. In that case, there is no redundancy in the topology axioms. I haven't seen this very often, however.

[2] It is interesting to note that, in the case of unions, this is similar to the convention that an empty sum is 0. In fact, if we assume that the unions are always disjoint and the subsets always finite, this similarity is a consequence of the addition principle in combinatorics.

*Proof.* Suppose for contradiction that there existed an $x \in \bigcup \varnothing$. Then, by definition, there exists a $B \in \varnothing$ such that $x \in B$. This is a contradiction, as $B \in \varnothing$ is impossible. Therefore $x$ cannot exist, and so it must be that $\bigcup \varnothing = \varnothing$.

Now, by definition, each element of $\bigcap \varnothing$ is an element of $X$, so $\bigcap \varnothing \subseteq X$. It therefore suffices to show that $X \subseteq \bigcap \varnothing$. Let $x \in X$. By definition, we need to show that for every $B \in \varnothing$ we have that $x \in B$. As $B \in \varnothing$ is never true, we have nothing to check and the condition holds vacuously. Therefore $x \in \bigcap \varnothing$, so $X \subseteq \bigcap \varnothing$ and $\bigcap \varnothing = X$. □

We can now show that the first axiom of a topology is redundant.

**Proposition.** *Let $X$ be a set and $\mathcal{T}$ be a collection of subsets of $X$. Suppose that $\mathcal{T}$ is closed under arbitrary unions and finite intersections. Then $\varnothing, X \in \mathcal{T}$.*

*Proof.* Since $\mathcal{T}$ is closed under arbitrary unions and $\varnothing \subseteq \mathcal{T}$, $\bigcup \varnothing \in \mathcal{T}$. By the previous proposition, we then have that $\varnothing \in \mathcal{T}$.

Since $\mathcal{T}$ is closed under finite unions and $\varnothing \subseteq \mathcal{T}$ is a finite set, $\bigcap \varnothing \in \mathcal{T}$. By the previous proposition, we then have that $X \in \mathcal{T}$. □

We can then define a topology on $X$ to be a collection of subsets of $X$ that is closed under arbitrary unions and finite intersections. The rest of the development of topology is relatively unaffected by this change, of course, since we can then immediately prove as we did above that every topology on $X$ contains $\varnothing$ and $X$.

The only place where I have seen this redundancy pointed out and then incoporated to make a more efficient definition of a topology is the nLab.